

A PERSPECTIVE ON THE CMB ACOUSTIC PEAK

T.A. MARRIAGE^{1,2}
Draft version February 1, 2008

ABSTRACT

CMB angular spectrum measurements suggest a flat universe. This paper clarifies the relation between geometry and the spherical harmonic index of the first acoustic peak (ℓ_{peak}). Numerical and analytic calculations show that ℓ_{peak} is approximately a function of Ω_K/Ω_M where Ω_K and Ω_M are the curvature ($\Omega_K > 0$ implies an open geometry) and mass density today in units of critical density. Assuming $\Omega_K/\Omega_M \ll 1$, one obtains $\ell_{peak} \approx \frac{11\sqrt{3}}{9(\sqrt{a_*+a_{eq}}-\sqrt{a_{eq}})} \left(2 + \frac{\Omega_K}{\Omega_M}\right)$ where a_* and a_{eq} are the scale factor at decoupling and radiation-matter equality. The derivation of ℓ_{peak} gives another perspective on the widely-recognized Ω_M - Ω_Λ degeneracy in flat models. This formula for near-flat cosmologies together with current angular spectrum data yields familiar parameter constraints.

Subject headings: cosmic microwave background — cosmological parameters

1. INTRODUCTION

It has long been recognized that the first peak in the CMB angular spectrum provides information about the curvature of the universe (Doroshkevich, Zel'dovich, & Syunyaev 1978; Bond et al. 1994; Kamionkowski, Spergel, & Sugiyama 1994; Efstathiou & Bond 1999; Cornish 2000; Weinberg 2000). The data are now in. TOCO, BOOMERanG, MAXIMA, and DASI have measured the peak position (Miller et al. 1999; Netterfield et al. 2001; Lee et al. 2001; Halverson et al. 2001). Analyses of the data strongly suggest a flat universe (Hu et al. 2001; Jaffe et al. 2000; Stompor et al. 2001; Pryke et al. 2001; Dodelson & Knox 2000). The MAP satellite, cosmic variance limited through $\ell \approx 600$, should make the definitive measurement in the near future (Page 2000).

As new data resolve higher order peaks, attention shifts to the physics at angular scales beyond that of the first maximum. The object of the present analysis is to clarify the physics derived from the position of the first peak, hereafter called “the peak index” or ℓ_{peak} . The size of the sound horizon r_{s*} at an angular diameter distance D_{a*} to decoupling determines the peak index.³ This is widely recognized and serves as a starting point. In Sections 2, 3, and 4, numerical and analytic calculations yield the peak index as a function of Ω_M and Ω_K . All results are checked with CMBFAST (Seljak & Zaldarriaga 1996). In Section 5, a simple formula for ℓ_{peak} , applicable to low Ω_K/Ω_M universes, is developed alongside geometric and classical interpretations of the formal results. It is found that the peak index approximates a function of Ω_K/Ω_M . Although our analysis grounds itself in the familiar concepts of D_{a*} and r_{s*} , the results are not widely recognized. While the physical effects responsible for ℓ_{peak} are understood, the interplay that gives the Ω_K/Ω_M dependence is not intuitive and holds a number of surprises. We end the investigation by using the functional form of ℓ_{peak} and current angular spectrum data to obtain parameter constraints re-

sembling those of more sophisticated treatments.

Throughout this work, the (Ω_M, Ω_K) plane serves as the parameter space. Other quantities enter, though the relation between ℓ_{peak} and Ω_K/Ω_M is relatively independent of them. To be concrete, we take $\Omega_B h^2 = 0.02$, consistent with nucleosynthesis (Burles, Nollett, & Turner 2001). The redshift of decoupling, z_* , is taken as 1400 from the Saha equation. The Hubble constant assumes 72 km/s/Mpc in accord with the HST Key Project results (Freedman et al. 2001).

2. ANGULAR DIAMETER DISTANCE

The comoving Friedmann-Robertson-Walker (FRW) metric for a closed universe ($\Omega_K < 0$) may be written as

$$ds^2 = d\eta^2 - R^2 (d\chi^2 + \sin^2 \chi d\Sigma^2) \quad (1)$$

where η denotes conformal time, $R^2 = -1/\Omega_K$, and $d\Sigma^2$ is the line element of a unit two-sphere. All spacetime intervals are given in units of H_0^{-1} . The angular diameter distance D_a to an object of proper length ds at comoving distance $R\chi$ ($d\eta = d\chi = 0$) is defined so that $d\Sigma = |ds|/D_a$.

$$D_a = R \sin \chi(a) \quad (2)$$

where χ^2 is the solution to the Friedmann equation:

$$d\chi^2 = \frac{-da^2}{\Omega_M(a/\Omega_K + a_{eq}/\Omega_K) + \Omega_\Lambda a^4/\Omega_K + a^2}. \quad (3)$$

Flat and open geometries are treated analogously.

Numerical Result D_{a*} can be evaluated numerically. The distance to redshift 1400 is computed for models across the (Ω_M, Ω_K) plane using cosmography routines from David Hogg (Hogg 1999).

Analytic Result To complement the numerical result, D_{a*} can be estimated analytically by setting $\Omega_\Lambda, a_{eq}, a_*$ to zero in equation (3), and then using Mattig’s solution (cf.

¹ University of Cambridge, Department of Applied Mathematics and Theoretical Physics, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA

² Current address: Princeton University, Department of Physics, Jadwin Hall, Princeton, NJ 08544

³ An expression subscripted by ‘*’ or ‘eq’ is evaluated at decoupling or matter-radiation equality respectively.

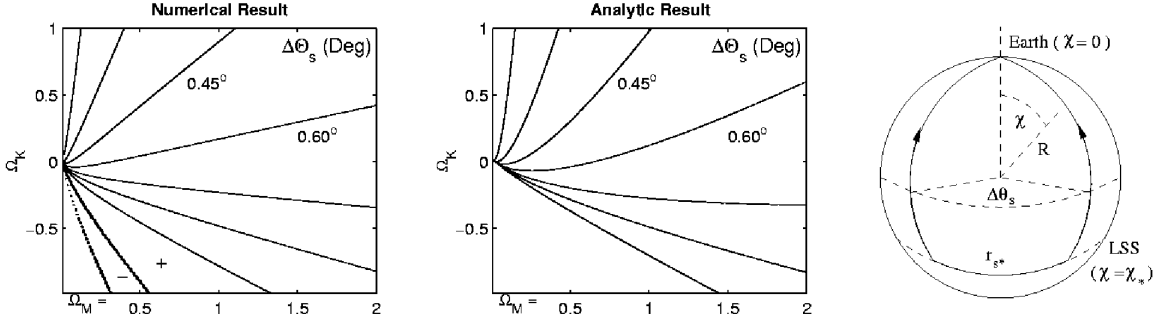


FIG. 1.— The angle subtended by the sound horizon at decoupling. $\Delta\theta_s = r_{s*}/D_{a*}$. Data favors $\Delta\theta_s = 0.60^\circ$ (Knox, Christensen, & Skordis 2001). In both numerical and analytic plots, equation (7) gives the sound horizon r_{s*} . In the numerical plot, the angular diameter distance to last scatter D_{a*} is computed using the cosmography code (Hogg 1999). In the analytic result, the classical equations for D_{a*} are integrated with $\Omega_\Lambda, a_{eq}, a_* = 0$, and the resulting formulae are used without assuming $\Omega_M + \Omega_K = 1$ (see eq. [4]). $\Delta\theta_s$ from either method approximates a function of Ω_K/Ω_M . The bending of the contours in the analytic result arises from radiation-dominated dynamics before last scatter. In the numerical result, explicit inclusion of Ω_Λ in the Friedmann equation balances the radiation effect: $\Delta\theta_s$ contours are straight lines converging on zero in the full calculation. At far right is illustrated the comoving coordinates of a closed FRW universe with time and azimuthal angular coordinates suppressed. Inspection of the drawing shows $\Delta\theta_s = r_{s*}/R \sin \chi_*$ where the comoving sound horizon at decoupling r_{s*} and the comoving distance to decoupling $R\chi_*$ are calculated using Friedmann dynamics.

(Peebles 1993)).

$$D_{a*} = \frac{2}{\sqrt{\Omega_M}} \times S(\gamma) \quad (4)$$

where $\gamma = 2 |\Omega_K| / \Omega_M$, and $S=1$ for a flat universe. If Ω_K is non-zero, then S takes the form

$$S(\gamma) = \sqrt{1/2\gamma} \times \begin{cases} \frac{1}{2}(\beta - 1/\beta) & : \Omega_K > 0 \\ \sqrt{1 - (\gamma - 1)^2} & : -\Omega_M < \Omega_K < 0 \end{cases}$$

$$\beta = \gamma + 1 + \sqrt{(\gamma + 1)^2 - 1}. \quad (5)$$

3. THE SOUND HORIZON

Sound travels through a tightly coupled photon-baryon system with speed c_s

$$c_s = \frac{1}{\sqrt{3(1+Q)}} \quad (6)$$

where Q is $3\rho_B/4\rho_\gamma$, and ρ_B and ρ_γ are the energy densities of baryons and radiation, respectively. The sound horizon at decoupling r_{s*} in comoving coordinates is then

$$r_{s*} = \int_{a=0}^{a_*} c_s \frac{d\eta}{da} da$$

$$= \frac{2}{\sqrt{3}\Omega_M} \sqrt{\frac{a_{eq}}{Q_{eq}}} \ln \frac{\sqrt{1+Q_*} + \sqrt{Q_* + Q_{eq}}}{1 + \sqrt{Q_{eq}}} \quad (7)$$

(Hu & Sugiyama 1996). Note that curvature does not affect sound dynamics before decoupling. In any geometry, r_{s*} will be the same.

4. THE HORIZON ANGLE AND THE PEAK INDEX

D_{a*} and r_{s*} give the angular size of the horizon:

$$\Delta\theta_s = \frac{r_{s*}}{D_{a*}}. \quad (8)$$

Values for $\Delta\theta_s$ corresponding to the numerical and analytic results for D_{a*} (see Section 2) are plotted in Figure 1. Curves of constant $\Delta\theta_s$ approximate straight lines, which

intersect the origin of the (Ω_M, Ω_K) plane. *The angle subtended by the sound horizon, and so the position of the first peak, approximates a function of Ω_K/Ω_M .* A well known corollary to this general statement is that a peak corresponding to $\Omega_K = 0$ should be insensitive to variations in Ω_M (Bond et al. 1994; Hu et al. 2001).

To check whether this simple analysis agrees with the standard model, we compare the above results to those from CMBFAST (Seljak & Zaldarriaga 1996). CMBFAST calculates the CMB angular spectrum.⁴ Then, ℓ_{peak} from the spectrum multiplies $\Delta\theta_s$ from Figure 1 to give the constant of proportionality $\alpha = \ell_{peak} \times \Delta\theta_s$. As shown in Figure 2, the numerically derived α increases from 110 ($\Omega_M = 0.14$) to 125 ($\Omega_M = 0.74$) and has weak if any dependence on curvature. The near constant graphs of α suggests a well-defined agreement between the peak indices from CMBFAST and those derived from this work's numerical and analytic calculations (see middle frame of Figure 2).

5. DISCUSSION

To gain intuition about $\Delta\theta_s$, consider the closed comoving universe (eq. [1]) illustrated in Figure 1, where time and azimuthal angular coordinates have been suppressed to produce the familiar two-sphere geometry. In this picture, CMB photons follow great circles, and, assuming a small angle, $\Delta\theta_s$ derives from inspection: $\Delta\theta_s = r_{s*}/R \sin \chi_*$, which is exactly equation (8).

To better understand the parameter dependence of r_{s*} , take the sound speed c_s (eq. [6]) to be constant at $9/10\sqrt{3}$. (With $\Omega_B h^2 = 0.02$ and radiation density derived from the COBE FIRAS measurement, the sound speed decreases at a near constant rate ($dc_s/da \approx \text{constant}$) from $c/\sqrt{3}$ at $a=0$ to four-fifths that value at decoupling (Fixen et al. 1997).) The sound horizon is then

$$r_{s*} = c_s \int_0^{a_*} \frac{d\eta}{da} da = \frac{9}{5\sqrt{3}\Omega_M} (\sqrt{a_* + a_{eq}} - \sqrt{a_{eq}}) \quad (9)$$

⁴ CMBFAST inputs are ($\Omega_B = 0.04, \Omega_\nu = 0, H_0 = 72, T_{cmb} = 2.726, Y_{He} = 0.24, N_\nu(\text{massless}) = 3.04, N_\nu(\text{massive}) = 0, \text{recfast, no reion, scalar only, primordial index 1, adiabatic}$).

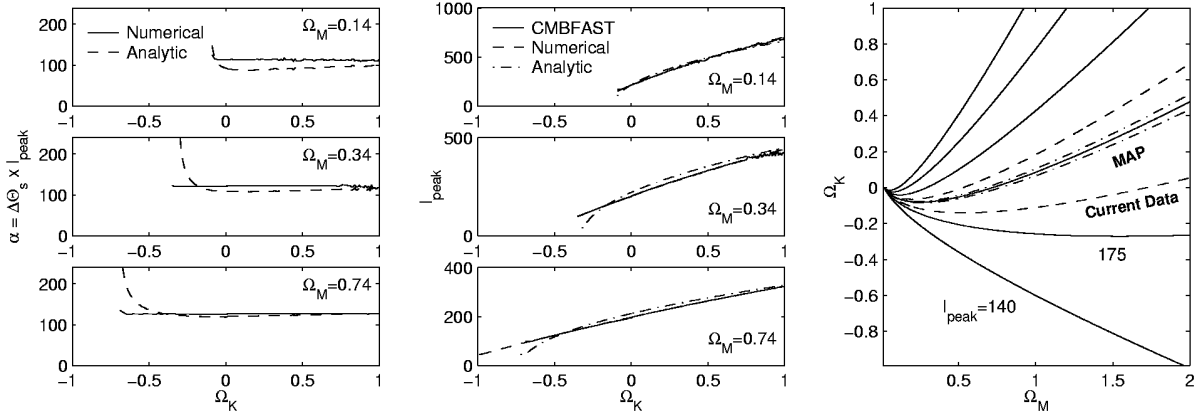


FIG. 2.— The relation $\alpha = \Delta\Theta_S \times \ell_{peak}$ and parameter constraints. The peak index from CMBFAST times $\Delta\Theta_S$ of numerical and analytic calculations gives α (left frame). The curves for α are averaged and reinserted into the equation $\ell_{peak} = \Delta\Theta_S/\alpha$ to generate the corresponding graphs of peak index (middle frame). Contours of ℓ_{peak} in the (Ω_M, Ω_K) plane are calculated using equations (9) and (10) (right frame). To coincide with the values of $\Delta\Theta_S$ and ℓ_{peak} derived from current data, the contour plot assumes $\alpha = 125 (\approx 0.6^\circ \times 210)$ throughout the plane (Knox, Christensen, & Skordis 2001; Knox & Page 2000; Hu et al. 2001). The dashed contours show constraints from current estimates of ℓ_{peak} and from the projected MAP satellite measurement.

where the final equality follows from the Friedmann equation (3) with $\Omega_K = \Omega_\Lambda = 0$.

To obtain a simple formula for ℓ_{peak} , expand D_{a*} to first order in $\gamma = 2 \mid \Omega_K \mid / \Omega_M$:

$$D_{a*} = \frac{1}{\sqrt{\Omega_M}} \left(2 + \frac{\Omega_K}{\Omega_M} \right). \quad (10)$$

The expansion applies to models with a low curvature-to-matter ratio. Coincidentally, such models are favored by experiment, so that the resulting formula for ℓ_{peak} is useful. Given equations (9) and (10) and a value of α from Figure 2,

$$\ell_{peak} = \frac{\alpha D_{a*}}{r_{s*}} \approx \frac{11\sqrt{3}}{9(\sqrt{a_* + a_{eq}} - \sqrt{a_{eq}})} \left(2 + \frac{\Omega_K}{\Omega_M} \right). \quad (11)$$

This near-flat approximation is plotted in Figure 2 where $a_{eq} \approx 2.4 \times 10^{-5} (\Omega_M h^2)^{-1}$. For flat models, the distance to last scatter (eq. [10]) scales as $\Omega_M^{-1/2}$. One may intuit that self-gravitation leads to smaller cosmological separations. This same “gravitational” effect decreases the distance between the big bang and last scatter, and therefore r_{s*} (eq. [9]) also scales with an overall factor of $\Omega_M^{-1/2}$. In equation (11), the $\Omega_M^{-1/2}$ dependence of D_{a*} cancels that of r_{s*} . This cancellation helps explain the Ω_M - Ω_Λ degeneracy of flat cosmogonies. Furthermore, the Ω_K/Ω_M term in equation (11) is proportional to the curvature times the area between light rays in a two-dimensional representation of an $\Omega_K \approx 0$ universe (e.g. Figure 1). This suggests that one may think of (Ω_M, Ω_K) dependence of ℓ_{peak} in near-flat spacetimes as resulting from light curving like the geodesics of a two-dimensional space of constant curvature. Finally, the effect of radiation on early universe dynamics and, in particular, on r_{s*} is made explicit by the appearance of a_{eq} in equation (9). Radiation-dominated cosmological growth per expansion scale ($d\eta/da$) is less

than that of matter-dominated dynamics. In low Ω_M universes, radiation brings last scattering even closer to the big bang and so shortens r_{s*} . Thus, the effect of radiation in the early universe is to spoil the pure Ω_K/Ω_M dependence of ℓ_{peak} as manifest by the curved contours in the plot of equation (11) in Figure 2. The straightness of the ℓ_{peak} contours is restored in the numerical result shown in Figure 1. This suggests that explicit inclusion of Ω_Λ in the computation of D_{a*} balances the a_{eq} dependence of r_{s*} .

6. CONCLUSION

The peak index has long been recognized as an indicator of geometry. It is hoped that the present analysis sheds new light on ℓ_{peak} . The peak index does not determine the magnitude of curvature, but rather the ratio of curvature to matter. A measurement of the peak’s angular scale gives the precise geometry only if $\Omega_K \approx 0$, otherwise ℓ_{peak} is a function of Ω_K/Ω_M . Furthermore, in deriving the Ω_K/Ω_M dependence of ℓ_{peak} , unexpected cosmological cancellings were discovered. Particularly useful is the balance of overall matter dependencies in r_{s*} and D_{a*} which helps account for the Ω_M - Ω_Λ degeneracy in flat models. At the same time, however, it is remarkable that the admittedly simple arguments of this work yield such a decisive cosmological indicator. Within the next few years, NASA’s MAP satellite data should give ℓ_{peak} to cosmic-variance levels. This measurement will burn a sharp line of possible worlds across the (Ω_M, Ω_K) plane.

This letter was begun at Cambridge as part of DAMTP’s tripos. TM thanks Ofer Lahav and Daniel Wesley for early guidance, David Hogg for cosmography computer code, and Lyman Page and James Peebles for suggestions regarding the final draft. TM is an NSF Graduate Research Fellow and is supported by NSF grant PHY-0099493.

REFERENCES

- Bond, J.R., Crittenden, R., Davis, R.L., Efstathiou, G., & Steinhardt, P.J. 1994, *Phys. Rev. Lett.*, 72, 13
- Burles, S., Nollett, K.M., & Turner, M.S. 2001, *ApJ*, 552, L1
- Cornish, N.J. 2000, preprint(astro-ph/0005261)
- Dodelson, S., & Knox, L. 2000, *Phys. Rev. Lett.*, 84, 3523
- Doroshkevich, A.G., Zel'dovich, Ya.B., & Syunyaev, R.A. 1978, *Sov. Astron.*, 22, 523
- Efstathiou, G., & Bond, J.R. 1999, *MNRAS*, 304, 75
- Fixen, D.J., Hinshaw, G., Bennet, C.L., and Mather, J.C. 1997, *ApJ*, 486, 623
- Freedman, W.L., et al. 2001, *ApJ*, 553, 47
- Halverson, N.W., et al. 2001, *ApJ*, submitted (astro-ph/0104489)
- Hogg, D.W. 1999, preprint(astro-ph/9905116)
- Hu, W., & Sugiyama, N. 1996, *ApJ*, 471, 542
- Hu, W., Fukugita, M., Zaldarriaga, M., & Tegmark, M. 2001, *ApJ*, 549, 669
- Jaffe, A.H., et al. 2000, *Phys. Rev. Lett.*, 86, 3475
- Kamionkowski, M., Spergel, D.N., & Sugiyama, N. 1994, *ApJ*, 426, L57
- Knox, L., & Page, L.A. 2000, *Phys. Rev. Lett.*, 85, 1366
- Knox, L., Christensen, N., & Skordis, C. 2001, *ApJ*, 563, L95
- Lee, A.T., et al. 2001, *ApJ*, 561, L1
- Miller, A.D., et al. 1999, *ApJ*, 524, L1
- Netterfield, C.B., et al. 2001, *ApJ*, in press (astro-ph/0104460)
- Page, L.A. 2000, in *Proc. IAU Symposium 201*, ed. A. Lasenby & A. Wilkinson, in press (astro-ph/0012214)
- Peebles, P.J.E. 1993, *Principles of Physical Cosmology*, (Princeton University Press)
- Pryke, C., Halverson, N.W., Leitch, E.M., Kovac, J., Carlstrom, J.E., Holzappel, W.L., & Dragovan, M. 2001, *ApJ*, submitted (astro-ph/0104490)
- Seljak, U., & Zaldarriaga, M. 1996, *ApJ*, 469, 437
- Stompor, R., et al. 2001, *ApJ*, 561, L7
- Weinberg, S. 2000, *Phys. Rev. D*, 62, 127302